

On a New Form of Quantum Mechanics

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We propose a new form of nonrelativistic quantum mechanics which is based on a quantum version of the action principle.

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I. INTRODUCTION

Quantum mechanics was originated in two principally different forms: the matrix mechanics of Heisenberg and the wave mechanics of Schrödinger. Both forms are equivalent in general and are the complement of one another. Latter Feynman [1] proposed a third formulation in terms of a path integral which is equivalent to the previous two. The use of any alternative formulation is obvious: it opens new possibilities in the development of the theory [2]. Following the logic of the quantum theory development one can suppose that not all alternative formulations are found. In this paper we propose a new formulation of the non-relativistic quantum mechanics with a quantum version of the action principle in a ground. We shall come to this formulation pushing from the Schrödinger wave theory. Then the canonical foundation of the quantum action principle will be done.

II. WAVE FUNCTIONAL

For simplicity we shall consider here the dynamics of a particle of mass m in the one-dimensional space. In the Schrödinger theory a quantum state of a particle is described by a wave function $\psi(x, t)$. This wave function has the meaning of the probability density to observe the particle nearby a point with a coordinate x at a moment of time t if the following normalization condition is fulfilled:

$$\int_{-\infty}^{+\infty} |\psi(x, t)|^2 dx = 1. \quad (1)$$

The evolution of a quantum state is described by the Schrödinger equation:

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\psi'' + U(x, t)\psi, \quad (2)$$

where the dot denotes the partial derivative on time and the hatch denotes the partial derivative on x -coordinate. The normalization condition (1) is conserved during the

Schrödinger evolution. The Schrödinger equation (2) may be obtained as the Euler-Lagrange equation for the action:

$$I_S[\psi] \equiv \int_0^T dt \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}i\hbar \left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi \right) - \frac{\hbar^2}{2m}\bar{\psi}'\psi' - U\bar{\psi}\psi \right]. \quad (3)$$

The wave function describes a quantum state of the particle at each moment of time. Let us introduce a new object – a function of a trajectory of the particle, i.e., a functional $\Psi[x(t)]$ which describes the particle dynamics on a whole time interval $[0, T]$. It will be called *the wave functional*. For this purpose, let us divide the interval $[0, T]$ by points t_n on N small parts of equal length $\varepsilon = T/N$. Let us approximate a trajectory $x = x(t)$ of the particle by a broken line with vertices $x_n = x(t_n)$. We define the wave functional as a product:

$$\Psi[x(t)] \equiv \prod_n \psi(x_n, t_n). \quad (4)$$

The normalization condition (1) may be rewritten as:

$$(\Psi, \Psi) \equiv \int \prod_n dx_n |\Psi[x(t)]|^2 = 1. \quad (5)$$

In fact, the wave functional is a function of $N + 1$ variables:

$$\Psi[x(t)] \equiv F(x_0, \dots, x_n, \dots, x_N), \quad (6)$$

but the limit $N \rightarrow \infty$ is assumed. Taking into account this limit, let us define the variation derivative of the functional (6) as follows:

$$\frac{\delta\Psi[x(t)]}{\delta x(t_n)} \equiv \frac{1}{\varepsilon} \frac{\partial F}{\partial x_n} = \frac{1}{\varepsilon} \frac{\partial \psi(x_n, t_n)}{\partial x_n} \prod_{n' \neq n} \psi(x_{n'}, t_{n'}). \quad (7)$$

Let us consider the integral defined by the corresponding integral sum:

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$$\int_0^T dt \dot{x}(t) \frac{\delta \Psi[x(t)]}{\delta x(t)} \cong \sum_n \Delta x_n \frac{1}{\varepsilon} \frac{\partial F}{\partial x_n}, \quad (8)$$

where $\Delta x_n \equiv x_{n+1} - x_n$. The integral (8) has the meaning of a variation of the wave functional (4) which is originated by an infinitesimal shift of the broken line “back” in time. This shift is represented as the successive replacement of vertices: $x_n \rightarrow x_{n+1}$. On the other hand, this variation may be considered as the result of the time shift on the one step back: $t \rightarrow t - \varepsilon$. Then one obtains:

$$\int_0^T dt \dot{x}(t) \frac{\delta \Psi[x(t)]}{\delta x(t)} \cong - \sum_n \frac{\partial \psi(x_n, t_n)}{\partial t} \prod_{n' \neq n} \psi(x_{n'}, t_{n'}). \quad (9)$$

III. QUANTUM ACTION PRINCIPLE

The action (3), which defines the Schrödinger dynamics in terms of the wave function $\psi(x, t)$, can be transformed in a functional of the wave functional $\Psi[x(t)]$. The following formula,

$$I_S[\psi] = - \int_0^T dt \int \prod_t dx \left\{ \frac{1}{2} i \tilde{\hbar} \dot{x}(t) \left[\bar{\Psi} \frac{\delta \Psi}{\delta x(t)} - \frac{\delta \bar{\Psi}}{\delta x(t)} \Psi \right] + \frac{\hbar^2}{2m} \frac{\delta \bar{\Psi}}{\delta x(t)} \frac{\delta \Psi}{\delta x(t)} + U(x(t), t) \bar{\Psi} \Psi \right\}, \quad (10)$$

is a “bridge” between the Schrödinger and our representations. For the multiplicative functional (4), which is related with the ε -division of the interval of time $[0, T]$, the constant $\tilde{\hbar}$ is

$$\tilde{\hbar} \equiv \varepsilon \hbar. \quad (11)$$

The proof of the formula (10) is based on the definition (7) of the variation derivative of the wave functional, the equation (9) and the normalization condition for the wave function (1) which is conserved in time.

The new representation of the Schrödinger action in terms of the wave functional gives us a possibility for an alternative formulation of quantum dynamics. The new formulation is based on the following equation:

$$\begin{aligned} \hat{I} \Psi &\equiv \int_0^T dt \left[\frac{\tilde{\hbar}}{i} \dot{x}(t) \frac{\delta \Psi}{\delta x(t)} + \frac{\tilde{\hbar}^2}{2m} \frac{\delta^2 \Psi}{\delta x^2(t)} - U(x(t), t) \Psi \right] \\ &= \lambda \Psi. \end{aligned} \quad (12)$$

Here λ is an eigenvalue of the operator \hat{I} which is a quantum version of the classical canonical action:

$$I[x(t), p(t)] \equiv \int_0^T dt \left[p \dot{x} - \frac{p^2}{2m} - U(x(t), t) \right]. \quad (13)$$

The “quantization” of the classical action (13) is performed by the replacement of the canonical momentum $p(t)$ by the functional-differential operator:

$$\hat{p}(t) \equiv \frac{\tilde{\hbar}}{i} \frac{\delta}{\delta x(t)}. \quad (14)$$

Let us formulate the quantum action principle as a search for an extremum in the set of eigenvalues λ , assuming that λ depends on a certain set of continuous parameters. According to (10), on the set of the multiplicative functionals (4) we have:

$$\lambda = (\Psi, \hat{I} \Psi) = I_S[\psi], \quad (15)$$

Therefore, the quantum action principle reduces to the well-known action principle of the Schrödinger wave mechanics if only multiplicative wave functionals are considered.

IV. NEW FORM OF CANONICAL QUANTIZATION

The canonical foundation of the new form of quantum dynamics consists in the definition of new rules of transition from classical to quantum mechanics. Old rules are a “quantum deformation” of classical dynamics which is formulated in the canonical form by use of the Poisson brackets (PB) [3]. These brackets are defined as the Lie brackets on the set of functions of canonical variables which obey the canonical relation (in the case of one dimension space):

$$\{x, p\} = 1, \quad (16)$$

see, for example, in Ref. [4]. The relation (16) is defined in a certain moment of time but the classical dynamics conserves PB-relations in time.

A modification of the classical dynamics proposed here consists in the replacement of the ordinary PB by a new ones which obey a non-simultaneous canonical relation (all others are equal to zero):

$$\{x(t), p(t')\} = \delta(t - t'). \quad (17)$$

The new definition of PB permits us to formulate classical equations of motion as the PB-relations:

$$\{x(t), I\} = \{p(t), I\} = 0, \quad (18)$$

where I is the classical action (13). The equations (18) are conditions of extremum of the classical action.

The canonical quantization of this theory consists in the replacement of classical canonical variables by operators and the replacement of canonical PB-relations by the corresponding commutation relations [3]. In our case the commutator

$$[\hat{x}(t), \hat{p}(t')] = i\tilde{\hbar}\delta(t - t') \quad (19)$$

defines a quantum algebra of the canonical variables, where $\tilde{\hbar}$ is a new “Plank” constant with the dimensionality $Dj \cdot s^2$. The canonical commutation relation (19) is valid if we consider, as usually, $\hat{x}(t)$ as the operator of product by $x(t)$ and $\hat{p}(t)$ as the operator of the functional differentiation (14) in the space of wave functionals

$\Psi[x(t)]$. After that, the quantum version of the action principle formulated in this work arises naturally.

V. CONCLUSIONS

The equation (12) is the main result of our work. It is an analog of the Schrödinger equation in the new formulation of quantum mechanics. The new formulation is equivalent to the Schrödinger theory, but it opens new possibilities for the development of quantum theory. The description of processes of birth and annihilation of particles without the use of the secondary quantization formalism will be one of the new possibilities.

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- [1] R. P. Feynman, *Rev. Mod. Phys.*, **20**, 367 (1948).
 - [2] R. P. Feynman, *Physics Today*, **19**, 31 (1966).
 - [3] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Yeshiva University, N.Y., 1964).

- [4] G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics* (W. A. Benjamin, Inc., N.Y., Amsterdam, 1963).